Schmidt's conjecture and Badziahin-Pollington-Velani's theorem

by Nikolay G. Moshchevitin[1]

Abstract. We give a simplified exposition of the easiest case of a breakthrough result by D.Badziahin, A.Pollington and S.Velani related to W.M.Schmidt's conjecture.

1. Schmidt's conjecture.

In this paper all numbers are real.

For $\alpha, \beta \in [0,1]$ under the condition $\alpha + \beta = 1$ and $\delta > 0$ we consider the sets

$$\mathrm{BAD}(\alpha, \beta; \delta) = \left\{ \xi = (\xi_1, \xi_2) \in [0,1]^2 : \inf_{p \in \mathbb{N}} \max\{p^\alpha ||p\xi_1||, p^\beta ||p\xi_2||\} \geqslant \delta \right\}$$

(here $|| \cdot ||$ denotes the distance to the nearest integer) and

$$\mathrm{BAD}(\alpha, \beta) = \bigcup_{\delta > 0} \mathrm{BAD}(\alpha, \beta; \delta).$$

In [1] Wolfgang M. Schmidt conjectured that for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$, $\alpha_1 + \beta_1 = \alpha_2 + \beta_2 = 1$ the intersection

$$\mathrm{BAD}(\alpha_1, \beta_1) \bigcap \mathrm{BAD}(\alpha_2, \beta_2)$$

is not empty. This conjecture was recently proved in a breakthrough paper by Dzmitry Badziahin, Andrew Pollington and Sanju Velani [2]. They proved a more general result: *for any finite collection of pairs* (α_j, β_j), $0 \leqslant \alpha_j, \beta_j \leqslant 1$, $\alpha_j + \beta_j = 1$, $1 \leqslant j \leqslant r$ *and for any* θ *under the condition*

$$\inf_{q \in \mathbb{N}} q ||q\theta|| > 0$$

the intersection

$$\bigcap_{j=1}^{r} \{\xi \in [0,1] : \ (\theta, \xi) \in \mathrm{BAD}(\alpha_j, \beta_j)\} \tag{1}$$

has full Hausdorff dimension.

Moreover one can take a certain infinite intersection in (1).

This result was obtained by an original method invented by D.Badziahin, A.Pollington and S.Velani. In the present paper we do not obtain any new result. The main purpose of the present paper is to give a more clear exposition of Badziahin-Pollington-Velani's method in the easiest case.

2. The simplest case.

The result by D.Badziahin, A.Pollington and S.Velani in the form (1) is non-trivial even for one set $\mathrm{BAD}(\alpha, \beta)$ and even for $\alpha = \beta = \frac{1}{2}$. In this case the result is as follows: *for* θ *such that*

$$\inf_{q \in \mathbb{N}} q^2 ||q\theta|| > 0 \tag{2}$$

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the set

$$\{\xi \in [0,1] : \ \ (\theta,\xi) \in \mathrm{BAD}(1/2,1/2)\}$$

has full Hausdorff dimension.

In the dual form the result proclaims that under the condition (2) the set

$$\{\xi \in [0,1] : \inf_{(A,B)\in\mathbb{Z}^2\setminus\{(0,0)\}} ||A\theta - B\xi|| \cdot \max(A^2,B^2) > 0\}$$

has full Hausdorff dimension.

In the present paper we show how Badziahin-Pollington-Velani's construction gives a proof of the following result.

Proposition 1. *Let*

$$0 < \delta \leqslant 2^{-1622}. \tag{3}$$

Suppose that

$$\inf_{q\in\mathbb{N}} q^2||q\theta|| \geqslant \delta. \tag{4}$$

Then there exists ξ *such that for all integers* A, B *with* $\max(|A|,|B|) > 0$ *one has*

$$||A\theta - B\xi|| \cdot \max(A^2,B^2) \geqslant \delta. \tag{5}$$

Of course the constant 2^{1622} in (3) may be reduced.

In sections 4 - 10 we give a complete proof of Proposition 1.

3. Acknowledgements.

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4. Parameters.

Suppose

$$R \geqslant 2^{422}$$

to be an integer. The integer parameter n increases to $+\infty$. Let

$$0 < \delta < \frac{1}{3R^{\frac{2533}{660}}}.$$

Put

$$\lambda = \frac{1741}{330}, \tag{6}$$

$$\kappa = \delta R^{\frac{6}{5}}, \tag{7}$$

so

$$\kappa \leqslant \frac{1}{3R^{\frac{\lambda}{2}}}. \tag{8}$$

Let k be an integer under the condition

$$1 \leqslant 2^k \leqslant R. \tag{9}$$

So

$$0 \leqslant k \leqslant \left\lfloor \frac{\log R}{\log 2} \right\rfloor. \tag{10}$$

Given k we define

$$d_k = \left\lfloor \left(\frac{\kappa}{\delta} \cdot \frac{2^k}{R} \right)^{\frac{2}{3}} \cdot R^{\frac{2}{165}} \right\rfloor \tag{11}$$

and

$$K_k = \frac{\delta}{\kappa} \cdot \frac{R^2}{2^k}. \tag{12}$$

One can easily see that

$$2 \leqslant \lfloor R^{\frac{8}{55}} \rfloor \leqslant d_k \leqslant R^{\frac{134}{165}} \tag{13}$$

and

$$d_k \cdot K_k \leqslant R^{\frac{52}{55}}. \tag{14}$$

5. Lines and forbidden intervals.

Given integers A, B, C with $(A, B, C) = 1, B > 0$ defne $L = L(A, B, C)$ to be a line

$$L = L(A, B, C) = \{(x, y) \in \mathbb{R}^2 : \ Ax - By + C = 0\}.$$

Put

$$H(A, B) = B \max(A^2, B^2). \tag{15}$$

Let

$$\Delta = \Delta(A, B, C) = \left(\frac{A\theta + C}{B} - \frac{\delta}{H(A, B)}, \frac{A\theta + C}{B} + \frac{\delta}{H(A, B)} \right)$$

be the interval of the length

$$|\Delta(A, B, C)| = \frac{2\delta}{H(A, B)}.$$

(Everywhere in the sequel $|J|$ stands for the length of an interval or a segment J.)

For our purpose it is enough to prove that

$$[0, 1] \setminus \left(\bigcup_{A,B,C} \Delta(A, B, C) \right) \neq \varnothing$$

where the union is taken over all triples of integers A, B, C such that

$$B > 0, \ \ (A, B, C) = 1.$$

It is convenient to consider the segment

$$\Theta = \{(x, y) \in \mathbb{R}^2 : \ x = \theta, \ 0 \leqslant y \leqslant 1\}.$$

Also it is convenient to consider intervals

$$\overline{\Delta} = \overline{\Delta}(A, B, C) = \{(x, y) \in \mathbb{R}^2 : \ x = \theta, \ y \in \Delta(A, B, C)\}.$$

So our task is to prove that

$$\Theta \setminus \left(\bigcup_{A,B,C} \overline{\Delta}(A,B,C) \right) \neq \varnothing. \tag{16}$$

6. Inductive construction.

We describe an inductive procedure to establish (16). We take an arbitrary segment

$$J_1 \subset \Theta$$

of the length

$$|J_1| = \frac{\kappa}{R}.$$

Now we describe the inductive process of constructing segments $J_n^\nu, n = 1, 2, 3, ...$

Given an integer $n \geqslant 1$ suppose we have a non-empty collection of segments

$$J_n^\nu \subset \Theta, \quad 1 \leqslant \nu \leqslant T_n, \tag{17}$$

$$|J_n^\nu| = \frac{\kappa}{R^n}$$

such that

$$J_n^\nu \cap \overline{\Delta}(A,B,C) = \varnothing \tag{18}$$

for all triples A, B, C under consideration such that $H(A,B) < R^{n-1}$. (For $n = 1$ this condition is empty.)

Each of the segments J_n^ν we divide into R equal segments

$$I_{n+1}^{\nu,\mu}, \quad 1 \leqslant \mu \leqslant R, \tag{19}$$

so

$$J_n^\nu = \bigcup_{1 \leqslant \mu \leqslant R} I_{n+1}^{\nu,\mu}, \quad |I_{n+1}^{\nu,\mu}| = \frac{|J_n^\nu|}{R} = \frac{\kappa}{R^{n+1}}.$$

We must consider the collection of the intervals

$$\overline{\Delta} = \overline{\Delta}(A,B,C), \tag{20}$$

$$R^{n-1} \leqslant H(A,B) < R^n \tag{21}$$

and prove that among the segments

$$I_{n+1}^{\nu,\mu}, \quad 1 \leqslant \mu \leqslant R, \ 1 \leqslant \nu \leqslant T_n \tag{22}$$

there exist a large number of segments $I_{n+1}^{\nu,\mu}$ such that

$$I_{n+1}^{\nu,\mu} \cap \overline{\Delta}(A,B,C) = \varnothing \tag{23}$$

for all intervals $\overline{\Delta}$ of the form (20) satisfying (21) .

In order to do this for any natural $m \leqslant n$ we must consider the corresponding collection of the segments

$$J_m^\nu \subset \Theta, \quad 1 \leqslant \nu \leqslant T_m, \quad |J_m^\nu| = \frac{\kappa}{R^m}$$

such that

$$J_m^\nu \cap \overline{\Delta}(A,B,C) = \varnothing$$

for all triples A, B, C under consideration such that $H(A,B) < R^{m-1}$.

Obviuosly $T_n > 0$ implies $T_m > 0$ for all $m \leqslant n$ as the collections are nested:

$$\bigcup_{1\leqslant \nu \leqslant T_1} J_1^\nu \supset \dots \supset \bigcup_{1\leqslant \nu \leqslant T_m} J_m^\nu \supset \bigcup_{1\leqslant \nu \leqslant T_{m+1}} J_{m+1}^\nu \dots \supset \bigcup_{1\leqslant \nu \leqslant T_n} J_n^\nu.$$

It happens that in order to show that many segments of the form (22) satisfy (23) we must assume that for any natural $m \leqslant n$ we have a certain lower bound for the quantity T_m. All precise estimates and inequalities will be formulated in the next sections.

7. Single interval $\overline{\Delta}(A, B, C)$.

Remind that the interval $\overline{\Delta}(A, B, C)$ has the length equal to $|\overline{\Delta}(A, B, C)| = \frac{2\delta}{H(A,B)}$. So given $\overline{\Delta}(A, B, C)$ the number of segments $I_{n+1}^{\nu,\mu}$ satistying

$$I_{n+1}^{\nu,\mu} \cap \overline{\Delta}(A, B, C) \neq \varnothing \tag{24}$$

is

$$\leqslant \frac{|\overline{\Delta}(A, B, C)|}{|I_{n+1}^{\nu,\mu}|} + 2 = \frac{2\delta}{\kappa} \cdot \frac{R^{n+1}}{H(A,B)} + 2. \tag{25}$$

Given k from the interval (9) consider the following condition on $H(A, B)$ which is stronger than the condition (21):

$$2^k R^{n-1} \leqslant H(A,B) = B \cdot \max(A^2, B^2) < \min(2^{k+1} R^{n-1}; R^n). \tag{26}$$

Let A, B satisfy the condition (26). Consider a fixed interval $\overline{\Delta}(A, B, C)$. We see (here we should refer to the definition (12) of the parameter K_k) that the the number of segments $I_{n+1}^{\nu,\mu}$ satisfying (24) with fixed A, B, C is less or equal than

$$2K_k + 2. \tag{27}$$

.

8. Lines with bounded coefficient $|A|/B$.

In this section we consider a single segment $J_n = J_n^\nu$ from the collection (17).

Given k from the interval (9) consider all the lines $L(A, B, C)$ such that coefficients A, B satisfy the condition (26) and the *additional* condition

$$B > R^{\frac{n}{3} - \lambda}. \tag{28}$$

The purpose of the current section is to prove that *the number of segments of the form (19) satisfying (24) for some interval* $\overline{\Delta}(A, B, C)$ *under conditions (26,28) is*

$$\leqslant \gamma R^{\frac{52}{55}}.$$

An admissible value for γ is $\gamma = 2^{13}$.

Recall that k satisfies (10). So from the desired upper bound for the number of segments satisfying (24) we see that *the number of segments of the form (19) satisfying (24) for some interval* $\overline{\Delta}(A, B, C)$ *under conditions (21,28) is*

$$\leqslant \gamma R^{\frac{52}{55}} \left(\left\lfloor \frac{\log R}{\log 2} \right\rfloor + 1 \right).$$

Note that under the conditions (26,28) one has

$$B \leqslant 2^{\frac{k+1}{3}} R^{\frac{n-1}{3}} \tag{29}$$

and

$$|A| \leqslant 2^{\frac{1}{6}} \cdot 2^{\frac{k+1}{3}} R^{\frac{n-1}{3}+\frac{\lambda}{2}}. \tag{30}$$

The last inequality follows from

$$A^2 \leqslant \frac{2^{k+1}R^{n-1}}{B} < 2^{k+1}R^{\frac{2(n-1)}{3}-\frac{1}{3}+\lambda} \leqslant 2^{\frac{1}{3}} \cdot 2^{\frac{2}{3}(k+1)} R^{\frac{2}{3}(n-1)+\lambda}.$$

Also from (26,28) we see that

$$\left(\frac{|A|}{B}\right)^2 \leqslant \frac{H(A,B)}{B^3} < \frac{R^n}{R^{n-3\lambda}} = R^{3\lambda}.$$

So

$$\frac{|A|}{B} \leqslant R^{\frac{3\lambda}{2}}. \tag{31}$$

8.1. Lemmata about lines intersecting a segment.

Here we give few lemmas. They will be useful not only in Section 8 but also in Section 9 where we consider a general situation.

Lemma 1. *Consider a segment $J_n = J_n^\nu$ from the collection (17). Suppose that there exist two lines*

$$L_1 = L(A_1, B_1, C_1), \;\; L_2 = L(A_2, B_2, C_2)$$

such that

$$L_i \cap J_n \neq \varnothing, \quad i = 1, 2$$

and

$$H(A_1, B_1), H(A_2, B_2) < R^n.$$

Then lines L_1 and L_2 are not parallel.

Proof. Lines $L_i, i = 1, 2$ intersect the segment $J_n \subset \Theta$ in points

$$\left(\theta, \frac{A_1\theta + C_1}{B_1}\right), \quad \left(\theta, \frac{A_2\theta + C_2}{B_2}\right)$$

with y-coordinates

$$\frac{A_1\theta + C_1}{B_1}, \quad \frac{A_2\theta + C_2}{B_2}, \quad \left|\frac{A_1\theta + C_1}{B_1} - \frac{A_2\theta + C_2}{B_2}\right| \leqslant |J_n|.$$

Suppose these lines to be parallel. Then

$$\frac{A_1}{B_1} = \frac{A_2}{B_2}$$

and

$$\frac{1}{B_1B_2} \leqslant \left|\frac{C_1}{B_1} - \frac{C_2}{B_2}\right| = \left|\frac{A_1\theta + C_1}{B_1} - \frac{A_2\theta + C_2}{B_2}\right| \leqslant |J_n| = \frac{\kappa}{R^n}.$$

From the inequality $B_j^3 \leqslant H(A_j, B_j) < R^n$ we see that $B_j < R^{\frac{n}{3}}$ and so

$$\frac{1}{B_1B_2} \geqslant \frac{1}{R^{\frac{2}{3}n}}.$$

As κ is small enough we have a contradiction. $\square$

Lemma 2. *Consider a segment $I \subset \{(x,y): \ x=\theta\}$ of the length $|I|$. Suppose that two lines*

$$L_1 = (A_1, B_1, C_1), \ \ L_2 = L(A_2, B_2, C_2)$$

intersect this segment I. Suppose that

$$L_1 \cap L_2 = P = \left(\frac{p}{q}, \frac{r}{q}\right), \ \ p, r, q \in \mathbb{Z}, \ \ q > 0, \ \ (p, r, q) = 1.$$

Then

(i) $|q\theta - p| \leqslant |I| B_1 B_2$;

(ii) $q \leqslant 2 \max(|A_1|, |A_2|) \max(B_1, B_2)$.

Proof. Obviously rational numbers $\frac{p}{q}, \frac{r}{q}$ satisfy

$$A_i \frac{p}{q} - B_i \frac{r}{q} + C_i = 0, \ \ i = 1, 2.$$

So

$$\frac{p}{q} = \frac{B_1 C_2 - B_2 C_1}{A_1 B_2 - A_2 B_1}, \ \ \frac{r}{q} = \frac{A_1 C_2 - A_2 C_1}{A_1 B_2 - A_2 B_1}.$$

As $(p, r, q) = 1$ we see that for some non-zero integer s one has

$$sq = A_1 B_2 - A_2 B_1,$$

$$sp = B_1 C_2 - B_2 C_1,$$

$$sr = A_1 C_2 - A_2 C_1.$$

So (ii) follows from the first of these three equalities as

$$q \leqslant |s| q = |A_1 B_2 - A_2 B_1| \leqslant 2 \max(|A_1|, |A_2|) \max(B_1, B_2).$$

Now

$$|q\theta - p| \leqslant |sq\theta - sp| = B_1 B_2 \cdot \left| \frac{A_1 \theta + C_1}{B_1} - \frac{A_2 \theta + C_2}{B_2} \right| \leqslant B_1 B_2 |I|,$$

and (i) follows. $\square$

Lemma 3. *All the lines $L = L(A, B, C)$ such that*

$$L(A, B, C) \cap J_n \neq \varnothing,$$

$$H(A, B) < R^n$$

satisfying the additional condition (28) have a single common point.

Proof.[2]

From Lemma 1 it follows that any two lines intersecting J_n have a common point. Suppose that we have three lines

$$L_i = L(A_i, B_i, C_i), \ \ \ i = 1, 2, 3$$

intersecting J_n which satisfy the conditions of Lemma 3 but do not have a common point. Then

$$D = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \neq 0.$$

[2]This proof was suggested to the author by Igor Rochev.

Let (θ, ξ) be the middle point of the segment J_n. Then

$$1 \leqslant |D| = | \begin{vmatrix} A_1 & B_1 & A_1\theta - B_1\xi + C_1 \\ A_2 & B_2 & A_2\theta - B_2\xi + C_2 \\ A_3 & B_3 & A_3\theta - B_3\xi + C_3 \end{vmatrix} |.$$

Suppose that $(\theta, Y_i) = J_n \cap L_i$. Then

$$Y_i = \frac{A_i\theta + C_i}{B_i}.$$

Now

$$|A_i\theta - B_i\xi + C_i| = B_i|\xi - Y_i| \leqslant B_i \cdot \frac{\kappa}{2R^n}.$$

Define

$$A = \max_{i=1,2,3} |A_i|, \;\; B = \max_{i=1,2,3} B_i.$$

Then

$$1 \leqslant |D| \leqslant 3 \cdot 2AB \cdot \frac{\kappa B}{2R^n} = \frac{3\kappa AB^2}{R^n}.$$

We have $B < R^{\frac{n}{3}}$ and $|A_i|^2 < \frac{R^n}{B_i} < R^{\frac{2}{3}n+\lambda}$. Recall that we suppose the condition (8) to be valid. So

$$1 \leqslant |D| < 3\kappa R^{\frac{\lambda}{2}} \leqslant 1.$$

This is not possible and lemma is proved.□

Lemma 4. *Consider two lines $L_i = L(A_i, B_i, C_i), i = 1, 2$. Suppose that A_1, B_1 satisfy the condition (26). Suppose that*

$$B_1 \geqslant B_2.$$

Let

$$L_1 \cap L_2 = P = \left(\frac{p}{q}, \frac{r}{q}\right).$$

Put

$$Y_i = \frac{A_i\theta + C_i}{B_i} \tag{32}$$

and suppose that

$$|Y_1 - Y_2| \leqslant \frac{|J_n|}{d_k} = \frac{\kappa}{d_k R^n}. \tag{33}$$

Put

$$\sigma_k = \frac{2\kappa}{d_k} \cdot \frac{2^k}{R} \cdot \frac{1}{|q\theta - p|}. \tag{34}$$

Then

$$B_1 \leqslant \sigma_k, \;\; A_1^2 \leqslant \sigma_k B_1. \tag{35}$$

Proof. From (33) it follows that lines L_1, L_2 intersect the line $\{(x, y) : \; x = \theta\}$ in points

$$\mathcal{Y}_1 = (\theta, Y_1,), \;\; \mathcal{Y}_2 = (\theta, Y_2) \tag{36}$$

where Y_i are defined in (32). We apply statement (i) of Lemma 2 with respect to the segment $I = [\mathcal{Y}_1, \mathcal{Y}_2]$ of the length $|I| \leqslant \frac{\kappa}{d_k R^n}$ to obtain the inequality

$$|q\theta - p| \leqslant \frac{\kappa}{d_k R^n} \cdot B_1 B_2 \leqslant \frac{\kappa}{d_k R^n} \cdot B_1^2.$$

From the condition (26) we see that

$$H(A_1,B_1)\leqslant 2^{k+1}R^{n-1},$$

and hence

$$R^n\geqslant \frac{1}{2}\cdot\frac{R}{2^k}\cdot H(A_1,B_1).$$

So

$$|q\theta-p|\leqslant \frac{2\kappa}{d_k}\cdot\frac{B_1^2}{H(A_1,B_1)}\cdot\frac{2^k}{R}$$

or

$$\max\left(\frac{A_1^2}{B_1},B_1\right)=\frac{H(A_1,B_1)}{B_1^2}\leqslant \frac{2\kappa}{d_k}\cdot\frac{2^k}{R}\cdot\frac{1}{|q\theta-p|}=\sigma_k$$

and Lemma 4 follows.□

8.2. Technical lemma.

In this section we prove a statement concerning the maximal value of the quantity $|A|/B$ under certain conditions.

Lemma 5. *Let $\sigma,W>0$. Suppose that real numbers A,B satisfy the following conditions:*

$$0<B\leqslant\sigma,\quad A^2\leqslant\sigma B,\quad H(A,B)\geqslant W.$$

Then

$$\frac{|A|}{B}\leqslant\left(\frac{\sigma^3}{W}\right)^{\frac{1}{4}}.$$

Proof. Obviously the maximal value of the ratio $|A|/B$ occurs at the point (A_*,B_*) which is a solution of the system

$$\begin{cases}A^2=\sigma B,\\ A^2B=W.\end{cases}$$

So

$$|A_*|=(\sigma W)^{\frac{1}{4}},\quad B_*=\left(\frac{W}{\sigma}\right)^{\frac{1}{2}},$$

and Lemma 5 follows. □

Collections $\mathfrak{A}$ and $\mathfrak{B}$.

Let

$$L_1,L_2,...,L_M \tag{37}$$

be all the lines $L(A,B,C)$ under conditions (26,28) intersecting the segment J_n. Suppose that $M\geqslant 2$. From Lemma 3 we know that all these lines pass through a single rational point

$$P=\left(\frac{p}{q},\frac{r}{q}\right)=\bigcap_{1\leqslant i\leqslant M}L_i.$$

Put

$$W_k=2^kR^{n-1},\ \ V_k=\left(\frac{\sigma_k^3}{W_k}\right)^{\frac{1}{4}}$$

(here σ_k is defined in (**??**)) and

$$\omega_k=\left|\theta-\frac{p}{q}\right|\cdot V_k=\left|\theta-\frac{p}{q}\right|\cdot\left(\frac{\sigma_k^3}{R^n}\frac{R}{2^k}\right)^{\frac{1}{4}}=\frac{(2\kappa)^{\frac{3}{4}}}{d_k^{\frac{3}{4}}qR^{\frac{n}{4}}}|q\theta-p|^{\frac{1}{4}}\left(\frac{2^k}{R}\right)^{\frac{1}{2}}. \tag{38}$$

We divide the collection of all the lines (37) into two subcollections $\mathfrak{A}$ and $\mathfrak{B}$.

Suppose that the collection $\mathfrak{A}$ consist of all lines of the form (37) that intersect the segment

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : \ x = \theta, \ y \in \left[\frac{r}{q} - \omega_k, \frac{r}{q} + \omega_k\right]\right\} \subset \{(x,y) : \ x = \theta\}. \tag{39}$$

Suppose that the collection $\mathfrak{B}$ consists of all lines of the form (37) that do not intersect the interval (39).

Lemma 6. *The number of elements in the collection $\mathfrak{B}$ is bounded by*

$$\#\mathfrak{B} \leqslant d_k.$$

Proof. Suppose that $\#\mathfrak{B} > d_k$. Then there exist two lines $L_1 = L(A_1, B_1, C_1), L_2 = L(A_2, B_2, C_2) \in \mathfrak{B}$ such that for the points (36) the inequality (33) is valid. Without loss of generality assume that $B_1 \geqslant B_2$. So we can apply Lemma 4 to see that A_1, B_1 satisfy inequalities (35). It means that A_1, B_1 satisfy the conditions of Lemma 5 with $\sigma = \sigma_k, W = W_k$. From Lemma 5 it follows that

$$\frac{|A_1|}{B_1} \leqslant V_k.$$

From the definition of ω_k we see that

$$Y_1 = \frac{A_1\theta + C_1}{B_1} \in \left[\frac{r}{q} - \omega_k, \frac{r}{q} + \omega_k\right].$$

So

$$L_1 \cap \Omega = (\theta, Y_1) \neq \varnothing.$$

It means that $L_1 \in \mathfrak{A}$. This is a contradiction. $\square$

In next two sections we deal with the collection $\mathfrak{A}$.

8.3. Collection $\mathfrak{A}$: the first principal inequality.

We suppose that

$$\#\mathfrak{A} \geqslant 2d_k. \tag{40}$$

Under this condition we deduce **the first principal inequality**:

Lemma 7. *Suppose that (40) is valid. Then*

$$qd_k \leqslant 12\sigma_k^2.$$

Proof. We divide the interval J_n into d_k intervals $J_n(\mu)$ of the equal length

$$|J_n(\mu)| = \frac{|J_n|}{d_k} = \frac{\kappa}{d_k R^n}.$$

Given interval $J_n(\mu)$ consider a single line $L = L(A,B,C) = L(\mu)$ from the collection (37) intersecting $J_n(\mu)$ and such that the coefficient B is the smallest one. (Of course such a line exists only in the case when the set of lines of the form (37) intersecting $J_n(\mu)$ is not empty.) So the number of lines $L(\mu)$ is bounded by the number of intervals $J_n(\mu)$, that is d_k. From Lemma 4 we see that for any line $L(A,B,C)$ from the collection $\mathfrak{A}$ different from lines $L(\mu)$ its coefficients must satisfy (35) and hence

$$\max(|A|, B) \leqslant \sigma_k. \tag{41}$$

We see that there exist $\geqslant d_k$ different lines from the collection $\mathfrak{A}$ with coefficients satisfying (41). Now we should make two observations.

1. All the lines from the collection $\mathfrak{A}$ pass through the rational point $P = \left(\frac{p}{q}, \frac{r}{q}\right)$. So the corresponding integer points (A, B) must belong to the lattice

$$\Lambda = \{(A, B) \in \mathbb{Z}^2 : \ \ Ap - Br \equiv 0 \pmod q\}$$

with the fundamental determinant

$$\det\Lambda = q.$$

2. As there is no parallel lines in the collection $\mathfrak{A}$ (Lemma 1) we see that the convex hull

$$\Pi = \mathrm{conv}\left(\{(0,0)\} \cup \{(A, B) : \ \exists C \ \ L(A, B, C) \in \mathfrak{A}\ , \ \ L(A, B, C) \neq L(\mu)\}\right)$$

is a polygon with positive measure $\mathrm{mes}\,\Pi$ (the last inequality takes into account that $d_k \geqslant 2$). We see that Π contains $> \#\mathfrak{A} - d_k \geqslant d_k$ points of the lattice Λ (here we make use of the condition (40)).

As the fundamental determinant of the lattice Λ is equal to q, by Pick's formula we have

$$\mathrm{mes}\,\Pi > \frac{q(\#\mathfrak{A} - d_k)}{6} \geqslant \frac{qd_k}{6}. \tag{42}$$

But from (41) it follows that

$$\Pi \subset \{(A, B) \in \mathbb{R}^2 : \ \ \max(|A|, B) \leqslant \sigma_k, \ \ B \geqslant 0\}.$$

So

$$\mathrm{mes}\,\Pi \leqslant 2\sigma_k^2. \tag{43}$$

Lemma 7 immediately follows from (42,43).$\square$

Lemma 8. *Under conditions of Lemma 7 one has*

$$|q\theta - p| \leqslant \frac{2\sqrt{12}\,\kappa}{d_k^{\frac{3}{2}} q^{\frac{1}{2}}} \cdot \frac{2^k}{R}.$$

Proof. Lemma 8 follows immediately from Lemma 7 and the definition of σ_k (equality (34)). $\square$

8.4. Interval Ω: the second principal inequality.

If $\mathfrak{A} \neq \varnothing$ then

$$\Omega \cap J_n \neq \varnothing. \tag{44}$$

This fact leads to the **second principal inequality**:

Lemma 9. *Suppose that under the conditions of Lemma 7 one has*

$$q < R^{\frac{2}{3}(n-1)}. \tag{45}$$

Then for the value ω_k defined in (38) one has

$$\omega_k \geqslant \frac{\delta}{2q^{\frac{3}{2}}}.$$

Proof.

We apply pigeonhole principle to see that there exist integers A, B, C such that $(A, B) \neq (0, 0)$ and

$$Ap - Br + Cq = 0$$

and

$$\max(|A|, B) \leqslant q^{\frac{1}{2}}, \quad B \geqslant 0. \tag{46}$$

In fact we prove that $B > 0$. Indeed if $B = 0$ then $A \neq 0$ and one has

$$|A\theta + C| = \left|A\left(\theta - \frac{p}{q}\right) + A\frac{p}{q} + C\right| = \left|A\left(\theta - \frac{p}{q}\right)\right| = \frac{|A|}{q}|q\theta - p| \leqslant \frac{|q\theta - p|}{q^{\frac{1}{2}}}.$$

From Lemma 8 and (11) we see that

$$|A\theta + C| \leqslant \frac{2\sqrt{12}\,\kappa}{d_k^{\frac{3}{2}} q} \cdot \frac{2^k}{R} < \frac{4\sqrt{12}\,\delta}{qR^{\frac{1}{55}}} \leqslant \frac{\delta}{q}$$

(as $R^{\frac{1}{55}} \geqslant 4\sqrt{12}$). But from (4) we see that

$$|A\theta + C| \geqslant \frac{\delta}{A^2} \geqslant \frac{\delta}{q}.$$

So we have a contradiction and hence $B > 0$.

Now

$$\left|A\theta - B\frac{r}{q} + C\right| = \left|A\left(\theta - \frac{p}{q}\right) + \frac{Ap - Br + Cq}{q}\right| = \left|A\left(\theta - \frac{p}{q}\right)\right| \leqslant \frac{|q\theta - p|}{q^{\frac{1}{2}}},$$

or

$$\left|\frac{A\theta + C}{B} - \frac{r}{q}\right| \leqslant \frac{|q\theta - p|}{Bq^{\frac{1}{2}}} \leqslant \frac{2\sqrt{12}\,\kappa}{B \cdot d_k^{\frac{3}{2}} q} \cdot \frac{2^k}{R}. \tag{47}$$

(here we apply Lemma 8 again).

The number $\frac{A\theta + C}{B}$ corresponds to the center of the interval

$$\overline{\Delta}(A, B, C)$$

with

$$H(A, B) \leqslant (\sqrt{q})^3 < R^{n-1}$$

(here we make use of (45,46)). From our inductive assumption in this situation one has

$$J_n \cap \overline{\Delta}(A, B, C) = \varnothing$$

(see (18)). Let

$$\mathcal{Y} = \left(\theta, \frac{A\theta + C}{B}\right)$$

be the center of the interval $\overline{\Delta}(A, B, C)$. One has

$$\operatorname{dist}(J_n, \mathcal{Y}) \geqslant \frac{|\overline{\Delta}(A, B.C)|}{2} = \frac{\delta}{H(A, B)} = \frac{\delta}{B \cdot \max(A^2, B^2)} \geqslant \frac{\delta}{Bq}. \tag{48}$$

Note that the point

$$\mathcal{Y}_* = \left(\theta, \frac{r}{q}\right)$$

is the center of the segment Ω. From (44) it follows that

$$\mathrm{dist}(J_n, \mathcal{Y}_*) \leqslant \omega_k. \tag{49}$$

Now we collect together (47,48,49) to see that

$$\omega_k + \frac{2\sqrt{12}\,\kappa}{B \cdot d_k^{\frac{3}{2}} q} \cdot \frac{2^k}{R} \geqslant \frac{\delta}{Bq}.$$

But as $R^{\frac{1}{55}} \geqslant 8\sqrt{12}$ we see that

$$\frac{2\sqrt{12}\,\kappa}{d_k^{\frac{3}{2}}} \cdot \frac{2^k}{R} \leqslant \frac{\delta}{2}.$$

So

$$\omega_k \geqslant \frac{\delta}{2Bq} \geqslant \frac{\delta}{2q^{\frac{3}{2}}}.$$

Lemma 9 is proved.□

8.5. The first fundamental lemma.

Fundamental Lemma 1. *Suppose we have a segment $J_n = J_n^\nu$ satisfying (18). Then the number of segments $I_{n+1}^{\nu,\mu}$ of the form (19) which has non-empty intersection with some interval*

$$\overline{\Delta}(A, B, C)$$

with A, B satisfying (21) and (28) is

$$\leqslant 2^{13} R^{\frac{52}{55}} \log R.$$

Proof.

1. Consider all values of parameter k for which $M < 3d_k$. For these k one can see that the number of lines from (37) intersecting J_n is less than $3d_k$. For each line from (37) the corresponding interval $\overline{\Delta}(A, B, C)$ can intersect not more than $2K_k + 2$ segments of the form (19). It may happen that a line $L(A, B, C)$ does not intersect the segment J_n but the corresponding interval $\overline{\Delta}(A, B, C)$ does intersect. But obviously such intervals can totally intersect not more than $2K_k + 2$ segments of the form (19). So for the parameter k under consideration the number of intersected segments of the form (19) is

$$\leqslant (2K_k + 2) \cdot 3d_k \leqslant 8R^{\frac{52}{55}}$$

(we take into account (13,14)).

2. Consider all values of parameter k for which $M \geqslant 3d_k$. In this case we have (40). So Lemma 8 gives the inequality

$$|q\theta - p| \leqslant \frac{2\sqrt{12}\,\kappa}{d_k^{\frac{3}{2}} q^{\frac{1}{2}}} \cdot \frac{2^k}{R}. \tag{50}$$

Recall that (38) gives

$$\omega_k = \frac{(2\kappa)^{\frac{3}{4}}}{d_k^{\frac{3}{4}} q R^{\frac{n}{4}}} |q\theta - p|^{\frac{1}{4}} \left(\frac{2^k}{R}\right)^{\frac{1}{2}},$$

and substituting here (50) we obtain

$$\omega_k \leqslant \frac{2 \cdot 12^{\frac{1}{8}} \kappa}{d_k^{\frac{9}{8}} q^{\frac{9}{8}} R^{\frac{n}{4}}} \cdot \left(\frac{2^k}{R} \right)^{\frac{3}{4}}. \tag{51}$$

From Lemma 9 we see that either

$$q \geqslant R^{\frac{2}{3}(n-1)}$$

or

$$\omega_k \geqslant \frac{\delta}{2q^{\frac{3}{2}}}.$$

From the last inequality and (51) we see that

$$q^{\frac{3}{8}} = q^{\frac{3}{2}-\frac{9}{8}} \geqslant \frac{1}{4 \cdot 12^{\frac{1}{8}}} \cdot \frac{\delta}{\kappa} \cdot d_k^{\frac{9}{8}} R^{\frac{n}{4}} \cdot \left(\frac{R}{2^k} \right)^{\frac{3}{4}}.$$

So in any case

$$q^{\frac{3}{8}} \geqslant \min \left(R^{\frac{1}{4}(n-1)}, \frac{1}{4 \cdot 12^{\frac{1}{8}}} \cdot \frac{\delta}{\kappa} \cdot d_k^{\frac{9}{8}} R^{\frac{n}{4}} \cdot \left(\frac{R}{2^k} \right)^{\frac{3}{4}} \right) = \frac{1}{4 \cdot 12^{\frac{1}{8}}} \cdot \frac{\delta}{\kappa} \cdot d_k^{\frac{9}{8}} R^{\frac{n}{4}} \cdot \left(\frac{R}{2^k} \right)^{\frac{3}{4}}$$

(to see that the minimum attains on the second element we take into account that the choice of parameters (9,11) shows that the first element in the minimum is greater than the second by the factor $R^{\frac{2}{55}}$). Substituting the last inequality into (51) we obtain

$$\omega_k \leqslant \frac{2^7 \sqrt{12} \kappa}{d_k^{\frac{9}{2}} R^n} \cdot \left(\frac{\kappa}{\delta} \cdot \frac{2^k}{R} \right)^3.$$

Now we must note that the number of segments of the form (19) which intersect with intervals $\overline{\Delta}(A, B, C)$ corresponding to the lines from the collection $\mathfrak{A}$ (recall that all the lines from the collection $\mathfrak{A}$ intersect the segment Ω of the length $2\omega_k$) is

$$\leqslant \frac{2\omega_k}{\kappa / R^{n+1}} + 2K_k + 2 \leqslant 2^{11} R^{\frac{52}{55}}$$

by (11,14).

As for the number of segments of the form (19) which intersect with intervals $\overline{\Delta}(A, B, C)$ corresponding to the lines from the collection $\mathfrak{B}$ we can say (Lemma 6) that this number is

$$\leqslant d_k (2K_k + 2) \leqslant 4R^{\frac{52}{55}}$$

by (14).

In the case 2 it may happen also that a line $L(A, B, C)$ does not intersect the segment J_n but the corresponding interval $\overline{\Delta}(A, B, C)$ does intersect. But obviously such intervals can totally intersect not more than $2K_k + 2$ segments of the form (19).

So the total number of segments of the form (19) which intersect with some intervals $\overline{\Delta}(A, B, C)$ under consideration is

$$\leqslant \frac{2\omega_k}{\kappa / R^{n+1}} + (d_k + 2)(2K_k + 2) \leqslant 2^{12} R^{\frac{52}{55}}.$$

Fundamental Lemma 1 follows as k takes its values in the interval $0 \leqslant k \leqslant \log R / \log 2$ (see (9)).$\square$

9. Lines with large coefficient $|A|/B$: parameter l.

Here we take an integer l such that

$$1 \leqslant l \leqslant \frac{n}{3\lambda}$$

and suppose that

$$R^{\frac{n}{3}-\lambda(l+1)} \leqslant B \leqslant R^{\frac{n}{3}-\lambda l} \tag{52}$$

In this section we consider a single segment $J_{n-l} = J_{n-l}^{\nu}$ from the collection (17) with fixed lower index $n-l$.

Let

$$J_n^{\nu}, \quad 1 \leqslant \nu \leqslant T$$

be *all* the segments such that

$$J_n^{\nu} \cap \overline{\Delta}(A,B,C) = \varnothing$$

for all triples A,B,C such that $H(A,B) < R^{n-1}$ and

$$J_n^{\nu} \subset J_{n-l}.$$

Each of the segments J_n^{ν} we divide into R smaller segments

$$I_{n+1}^{\nu,\mu}, \ \ 1 \leqslant \nu \leqslant T, \ \ 1 \leqslant \mu \leqslant R \tag{53}$$

of equal length

$$|I_{n+1}^{\nu,\mu}| = \frac{|J_n^{\nu}|}{R} = \frac{\kappa}{R^{n+1}}.$$

such that

$$J_n^{\nu} = \bigcup_{1\leqslant\mu\leqslant R} I_{n+1}^{\nu,\mu}, \quad 1 \leqslant \nu \leqslant T.$$

The purpose of the current section is to prove that *the number of segments of the form (53) satisfying*

$$I_{n+1}^{\nu,\mu} \cap \overline{\Delta}(A,B,C) \neq \varnothing \tag{54}$$

for some interval $\overline{\Delta}(A,B,C)$ with coefficients A,B satisfying the conditions (21) and satisfying the additional condition (52) is

$$\leqslant \gamma_1 R^{\frac{52}{55}}.$$

An admissible value for γ_1 is $\gamma_1 = 8$.

Under the conditions (21,52) one has

$$|A| \leqslant R^{\frac{n}{3}+\frac{\lambda(l+1)}{2}} \tag{55}$$

and

$$\frac{|A|}{B} \leqslant R^{\frac{3}{2}\lambda(l+1)}. \tag{56}$$

In the rest part of this section we modify lemmas 1 - 4 and 9 is the case of the inequalities (52). Proofs of all lemmas below are quite similar to the proofs of lemmas behind.

9.1. Modified lemmata about lines intersecting a segment.

Lemma 1*. *Consider a segment $J_{n-l} = J_{n-l}^{\nu}$. Suppose that there exist two lines*

$$L_1 = L(A_1,B_1,C_1), \ \ L_2 = L(A_2,B_2,C_2)$$

such that

$$L_i \cap J_{n-l} \neq \varnothing, \quad i = 1, 2$$

and

$$H(A_1, B_1), H(A_2, B_2) < R^n.$$

Then lines L_1 and L_2 are not parallel.

Proof. Lines $L_i, i = 1, 2$ intersect the segment $J_{n-l} \subset \Theta$ in points

$$\left(\theta, \frac{A_1\theta + C_1}{B_1}\right), \quad \left(\theta, \frac{A_2\theta + C_2}{B_2}\right)$$

with y-coordinates

$$\frac{A_1\theta + C_1}{B_1}, \quad \frac{A_2\theta + C_2}{B_2}, \quad \left|\frac{A_1\theta + C_1}{B_1} - \frac{A_2\theta + C_2}{B_2}\right| \leqslant |J_{n-l}|.$$

Suppose these lines to be parallel. Then

$$\frac{A_1}{B_1} = \frac{A_2}{B_2}$$

and by making use of (52) we have

$$\frac{\kappa}{R^{n-l}} < \frac{R^{\frac{n}{3}+(2\lambda-1)l}}{R^{n-l}} = \frac{R^{2\lambda l}}{R^{\frac{2n}{3}}} \leqslant \frac{1}{B_1 B_2} \leqslant \left|\frac{C_1}{B_1} - \frac{C_2}{B_2}\right| = \left|\frac{A_1\theta + C_1}{B_1} - \frac{A_2\theta + C_2}{B_2}\right| \leqslant |J_{n-l}| = \frac{\kappa}{R^{n-l}}$$

(here we use the inequality $\kappa < 1 < R^{\frac{n}{3}+(2\lambda-1)l}$) and this is a contradiction. $\Box$

We do not need any changes in Lemma 2. But in the case $l \geqslant 1$ simple application of Lemma 1 gives a strong inequality. This inequality we formulate as

Lemma 2* *Suppose that two lines*

$$L_1 = L(A_1, B_1, C_1), \;\; L_2 = L(A_2, B_2, C_2), \quad L_i \cap J_{n-l} \neq \varnothing, \quad i = 1, 2$$

satisfy

$$H(A_i, B_i) \leqslant R^n, \quad i = 1, 2.$$

Suppose the additional condition (52) to be valid. Then

$$|q\theta - p| \leqslant \kappa R^{-\frac{n}{3}-(2\lambda-1)l}.$$

Proof. We should take in Lemma 2 $I = J_{n-l}$ and combine the conclusion (i) with (52).$\Box$

Lemma 3*. *All the lines $L = L(A, B, C)$ such that*

$$L(A, B, C) \cap J_{n-l} \neq \varnothing,$$

$$H(A, B) < R^n$$

satisfying the addditional condition (52) have a single common point.

Proof.

The proof is quite close to the proof of Lemma 3. From Lemma 1* it follows that any two lines intersecting J_{n-l} have a common point. Suppose that we have three lines

$$L_i = L(A_i, B_i, C_i), \quad i = 1, 2, 3$$

intersecting J_{n-l} which satisfy the conditions of Lemma 3* but do not have a common point. Then (by taking $(\theta,\ \xi)$ to be the middle of J_{n-l} and $(\theta, Y_i) = J_{n-l} \cap L_i$ we see that

$$|A_i\theta - B_i\xi + C_i| = B_i|\xi - Y_i| \leqslant B_i \cdot \frac{\kappa}{2R^{n-l}}$$

for every $i = 1, 2, 3$) we have

$$1 \leqslant |D| = | \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} | \leqslant 3 \cdot 2AB \cdot \frac{\kappa B}{2R^{n-l}} = \frac{3\kappa AB^2}{R^{n-l}},$$

where

$$A = \max_{i=1,2,3} |A_i| < R^{\frac{n}{3}+\frac{(l+1)\lambda}{2}},\ \ B = \max_{i=1,2,3} B_i \leqslant R^{\frac{n}{3}-\lambda l}$$

(the first inequality here follows from inequalities (52) as $|A_i|^2 \leqslant \frac{R^n}{B_i} \leqslant \frac{R^n}{R^{\frac{n}{3}-(l+1)\lambda}} = R^{\frac{2}{3}n+(l+1)\lambda}$).

Recall that we suppose the condition (8) to be valid and λ satisfies (6). So

$$1 \leqslant |D| < 3\kappa R^{\frac{(l+1)\lambda}{2}-(2\lambda-1)l} < 1.$$

This is not possible and lemma is proved.□

Now we suppose that all the lines intersecting the segment J_{n-l} and satisfying $H(A, B) < R^{n-1}$ and the additional condition (52) pass through a single point

$$P = \left(\frac{p}{q}, \frac{r}{q}\right).$$

Put

$$\sigma(l) = \frac{\kappa R^l}{|q\theta - p|}. \tag{57}$$

Lemma 4*. *Consider two lines $L_i = L(A_i, B_i, C_i), i = 1, 2$. Suppose that A_1, B_1, A_2, B_2 satisfy*

$$H(A_1, B_1), H(A_2, B_2) < R^n.$$

Suppose that both lines L_1, L_2 intersect the segment J_{n-l}. Suppose that

$$B_1 \geqslant B_2.$$

Then with $\sigma(l)$ defined in (57) one has

$$B_1 \leqslant \sigma(l),\ \ A_1^2 \leqslant \sigma(l)\, B_1. \tag{58}$$

Proof. By Lemma 2 (statement (i)) we have

$$|q\theta - p| \leqslant \frac{\kappa}{R^{n-l}} \cdot B_1B_2 \leqslant \frac{\kappa}{R^{n-l}} \cdot B_1^2.$$

As in Lemma 4 we see that

$$\max\left(\frac{A_1^2}{B_1}, B_1\right) = \frac{H(A_1, B_1)}{B_1^2} \leqslant \frac{\kappa}{|q\theta - p|} \cdot R^l = \sigma(l)$$

and Lemma 4^* follows.□

Put

$$V(l) := \left(\frac{(\sigma(l))^3}{R^{n-1}}\right)^{\frac{1}{4}}. \tag{59}$$

Corollary 1. *Suppose that the conditions of Lemma 4* are satisfied and in addition we have (21). Then*

$$\frac{|A_1|}{B_1} \leqslant V(l). \tag{60}$$

Proof. Apply Lemma 5 with $\sigma = \sigma(l), W = R^{n-1}$.□

Corollary 2. *Let*

$$L_1, L_2, ..., L_M, \quad L_j = L(A_j, B_j, C_j) \tag{61}$$

be all the lines intersecting J_{n-l} and satisfying (21). Then for all j from the interval $1 \leqslant j \leqslant M$ but one possible exception one has

$$\frac{|A_j|}{B_j} \leqslant V(l). \tag{62}$$

Proof. Among the collection (61) we have a line with the minimal coefficient B_j. By (60) of Corollary 1 we see that all other lines satisfy (62). □

9.2. Collections $\mathfrak{A}_l$ and $\mathfrak{B}_l$.

In the sequel we suppose that $M \geqslant 2$. We divide the collection of lines (61) into two subcollections. Collection $\mathfrak{B}_l$ consists of only one line with the minimal value of B. So

$$\#\mathfrak{B}_l = 1. \tag{63}$$

All other lines form the collection $\mathfrak{A}_l$. By the arguments form the proof of Corolary 2 we see that for any L_j from the collection $\mathfrak{A}_l$ we have (62). So all these lines intersect the segment Θ in the points of the segment

$$\Omega(l) = \left[\frac{r}{q} - \omega(l), \frac{r}{q} + \omega(l)\right],$$

where

$$\omega(l) = \left|\theta - \frac{p}{q}\right| \cdot V(l) = \frac{|q\theta - p|(\sigma(l))^{\frac{3}{4}}}{qR^{\frac{n-1}{4}}} = \frac{\kappa^{\frac{3}{4}}|q\theta - p|^{\frac{1}{4}} R^{\frac{3l+1}{4}}}{qR^{\frac{n}{4}}} \tag{64}$$

by the definitions of $\sigma(l)$ and $V(l)$ (see (57,59)). We apply Lemma 2^* to deduce from (64) the inequality

$$\omega(l) \leqslant \frac{\kappa}{qR^{\frac{n}{3}}} \cdot R^{-\frac{\lambda l}{2} + \frac{4l+1}{4}}. \tag{65}$$

9.3. Collection $\mathfrak{A}_l$: lower bound for q and its application.

We deal with the situation $l \geqslant 1$. In this case the consideration of the collection $\mathfrak{A}_l$ is much more simple. The only thing what we need is an analog of Lemma 9 and its corollary for the lower bound of q.

Lemma 9^*. *Suppose that*

$$q < R^{\frac{2}{3}(n-l-1)}. \tag{66}$$

Then for the value $\omega(l)$ defined in (64) one has

$$\omega(l) \geqslant \frac{\delta}{2q^{\frac{3}{2}}}.$$

Proof. Similarly to the proof of Lemma 9 we find integers A, B, C such that $(A, B) \neq (0,0)$ and

$$Ap - Br + Cq = 0, \ \max(|A|, B) \leqslant q^{\frac{1}{2}}, \ \ B \geqslant 0.$$

Then

$$\left|A\theta - B\frac{r}{q} + C\right| = \left|A\left(\theta - \frac{p}{q}\right) + \frac{Ap - Br + Cq}{q}\right| = \left|A\left(\theta - \frac{p}{q}\right)\right| \leqslant \frac{|q\theta - p|}{q^{\frac{1}{2}}},$$

From the condition (66) we have

$$R^{\frac{n}{3}} \geqslant q^{\frac{1}{2}} R^{\frac{l+1}{3}}.$$

So we take into account Lemma 2^* to see that

$$\left|A\theta - B\frac{r}{q} + C\right| \leqslant \frac{|q\theta - p|}{q^{\frac{1}{2}}} \leqslant \frac{\kappa R^{-\frac{n}{3}-(2\lambda-1)}}{q^{\frac{1}{2}}} \leqslant \frac{\kappa}{q} \cdot R^{-(2\lambda-1)-\frac{l+1}{3}}.$$

As

$$\frac{\delta}{q} \leqslant \frac{\delta}{A^2} \leqslant |A\theta + C|$$

by (4) and $\delta > \kappa \cdot R^{-(2\lambda-1)-\frac{l+1}{3}}$ we have $B > 0$. As

$$\max(|A|, B) < R^{\frac{n-l-1}{3}}$$

it follows that

$$\overline{\Delta}(A, B, C) \cap J_{n-l} = \varnothing.$$

By following all the arguments of the proof of Lemma 9 we see that

$$\omega(l) + \frac{\kappa}{Bq} \cdot R^{-(2\lambda-1)-\frac{l+1}{3}} \geqslant \frac{\delta}{Bq}.$$

As $\lambda > 3$ and $\kappa = \delta R^{\frac{6}{5}}$ we have

$$\frac{\delta}{2} > \kappa \cdot R^{-(2\lambda-1)-\frac{l+1}{3}}.$$

Lemma 9 * follows.□

Corollary 1. *The following inequality is valid:*

$$q \geqslant R^{\frac{2}{3}(n-l-1)}.$$

Proof. Suppose that (66) is valid. Then by Lemma 9^* we have

$$\omega(l) \geqslant \frac{\delta}{2q^{\frac{3}{2}}}.$$

Combining this inequality with (65) we have

$$\frac{\delta}{2q^{\frac{3}{2}}} \leqslant \frac{\kappa}{qR^{\frac{n}{3}}} \cdot R^{-\frac{\lambda l}{2}+\frac{4l+1}{4}}.$$

Hence

$$q \geqslant \frac{1}{4}\left(\frac{\delta}{\kappa}\right)^2 R^{\frac{2}{3}n+\lambda l-\frac{4l+1}{2}} = \frac{1}{4} \cdot R^{\frac{2}{3}n+(\lambda-2)l-\frac{29}{10}} > R^{\frac{2}{3}(n-l-1)},$$

as $\lambda > 4$ and
$$(\lambda-2)l-\frac{29}{10} > 2l-\frac{29}{10} \geqslant -\frac{9}{10} > -\frac{4}{3} \geqslant \frac{2}{3}(-l-1).$$
Corollary 1 is proved.$\square$

Corollary 2. *In the case $l \geqslant 1$ we have the following upper bound:*
$$\omega(l) \leqslant \frac{\kappa}{R^n}\cdot R^{-\frac{\lambda l}{2}+\frac{20l+11}{12}}.$$

Proof. Apply (65) and the inequality of Corollary 1.$\square$

9.4. The second fundamental lemma.

Here we prove the following

Fundamental Lemma 2. *Let $l \geqslant 1$. Suppose we have a segment J_{n-l}. Then the number of segments $I_{n+1}^{\nu,\mu}$ of the form (53) which intersect with some interval*
$$\overline{\Delta}(A, B.C)$$
with A, B satisfying (21) and (52) is
$$\leqslant 8R^{\frac{52}{55}}.$$

Proof.

First of all we suppose that $M \geqslant 2$ (otherwise there exists only one line L_1 under consideration and we may use the san=me arguments as for the collection $\mathfrak{B}_l$, see below).

1. Lines from the collection $\mathfrak{A}_l$ intersect the segment Θ. The points of intersection belong to the segment Ω_l of the length $2\omega(l)$ satisfying upper bound (65). For $L(A,B,C) \in \mathfrak{A}_l$ one has
$$|\overline{\Delta}(A,B,C)| = \frac{2\delta}{H(A,B)} \leqslant \frac{2\delta}{R^{n-1}}.$$

So the number of segments $I_{n+1}^{\nu,\mu}$ of the form (53) which intersect with intervals $\overline{\Delta}(A,B,C)$ corresponding to the collection $\mathfrak{A}_l$ is less or equal than
$$\frac{2\omega(l)+\max|\overline{\Delta}(A,B,C)|}{\kappa/R^{n+1}}+2 \leqslant 2R^2\cdot\frac{\delta}{\kappa}+2R^{-\frac{\lambda l}{2}+\frac{20l+23}{12}}+2 \leqslant 2R^2\cdot\frac{\delta}{\kappa}+2R^{-\frac{\lambda}{2}+\frac{43}{12}}+2 \leqslant 4R^{\frac{52}{55}}+2$$
as $\frac{\delta}{\kappa} = R^{-\frac{6}{5}}$ and $\lambda = \frac{1741}{330}$.

2. The number of segments $I_{n+1}^{\nu,\mu}$ of the form (53) which intersect with intervals $\overline{\Delta}(A,B,C)$ corresponding to the collection $\mathfrak{B}_l$ is less or equal than
$$\frac{2\delta/R^{n-1}}{\kappa/R^{n+1}}+2 \leqslant 2R^{\frac{4}{5}}+2.$$

Also we must take into account that a line $L(A,B,C)$ may not intersect the segment J_{n-l} but the corresponding interval $\overline{\Delta}(A,B,C)$ may intersect it. But obviously such intervals can totally intersect not more than $2R^{\frac{4}{5}}+2$ segments of the form (53).

The second Fundamental Lemma follows.$\square$

10. Proof of Proposirion 1.

We apply Fundamental Lemmas 1 and 2. Arguments below are close to those from Peres-Schlag's method (see [3]).

Recall that we denote by T_n the total number of segments J_n^ν.

By Fundamental Lemmas 1,2 we see that

$$T_{n+1} \geqslant T_n \cdot R - T_n \cdot 2^{13} R^{\frac{52}{55}} \log R - \sum_{l=1}^{[n/3\lambda]} T_{n-l} \cdot 8R^{\frac{52}{55}}$$

or

$$T_{n+1} \geqslant T_n \left(R - 2^{13} R^{\frac{52}{55}} \log R - \sum_{l=1}^{[n/3\lambda]} \frac{T_{n-l}}{T_n} \cdot 8R^{\frac{52}{55}} \right).$$

We see by induction that

$$T_{n+1} \geqslant T_n \cdot (R - 2^{14} R^{\frac{52}{55}} \log R)$$

or

$$T_n \geqslant (R - 2^{14} R^{\frac{52}{55}} \log R)^{n-1}.$$

In fact as $R \geqslant 2^{422}$ this inequality proves that $T_n > 0$ for every n. It means that

$$\bigcap_{n \in \mathbb{N}} \bigcup_{1 \leqslant \nu_n \leqslant T_n} J_n^{\nu_n} \neq \varnothing.$$

By putting $R = 2^{422}$ we prove Proposition 1. $\square$

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author: Nikolay G. Moshchevitin;
e-mail: moshchevitin@rambler.ru